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The validity of the Background Field Approximation

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Abstract

In the absence of a tractable theory of quantum gravity, quantum matter field effects have been so far computed by treating gravity at the Background Field Approximation. The principle aim of this paper is to investigate the validity of this approximation which is not specific to gravity. To this end, for reasons of simplicity and clarity, we shall compare the descriptions of thermal processes induced by constant acceleration (i.e. the Unruh effect) in four dynamical frameworks. In this problem, the position of the "heavy" accelerated system plays the role of gravity. In the first framework, the trajectory is treated at the BFA: it is given from the outset and unaffected by radiative processes. In the second one, recoil effects induced by these emission processes are taken into account by describing the system's position by WKB wave functions. In the third one, the accelerated system is described by second quantized fields and in the fourth one, gravity is turned on. It is most interesting to see when and why transitions amplitudes evaluated in different frameworks but describing the same process do agree. It is indeed this comparison that determines the validity of the BFA. It is also interesting to notice that the abandonment of the BFA delivers new physical insights concerning the processes. For instance, in the fourth framework, the "recoils" of gravity show that the acceleration horizon area acts as an entropy in delivering heat to accelerated systems.

1 Introduction

In order to analyze the stability of a given configuration, one must choose a framework in which the dynamical consequences of certain class of fluctuations may be investigated. It is quite obvious that the stability conditions might drastically depend on the class of fluctuations considered. Moreover, in view of the complexity of most physically interesting situations, certain simplifying approximations must be made. These in turn specify the particular dynamical framework and may therefore restrict the class of fluctuations one is effectively taken into account. Because of this, stability conditions might also depend on the nature of the approximations used. In particular, upon dealing with gravitational systems, the absence of any tractable quantum theory of gravity imposes to treat gravity at the Background Field Approximation (BFA) even when one considers quantum effects induced by quantum matter fields. This implies that one can only compute the gravitational response induced by the mean (i.e. quantum average) energy-momentum tensor. One must thus inquire onto the validity of the results obtained by using the BFA for describing gravity. In particular, when one abandons the BFA, does one obtain a behavior more or less singular ?

These considerations certainly apply to unstable or near unstable situations. Examples are provided by matter and gravity dynamics near inner horizons of charged black holes[1][2] as well as the dynamics near the event horizon of extremal black holes[3][4]. However, they might also apply to (apparently) more stable situations, such as, for instance, the evaporation process of Schwarzschild holes[5]. This is because the whole dynamical evolution might drastically depend on the nature of the approximations. Indeed, when working in the semi-classical approximation, one only deals with regular mean energy densities[6][7][8]. However, Hawking quanta issue, in usual quantum field theory, from vacuum fluctuations characterized by ultra-high frequencies (trans-Planckian), see [9]-[12]. These fluctuations give rise to trans-Planckian matrix elements of the stress energy-tensor[13][14]. Therefore, since these matrix elements should determine (yet unknown) quantum gravitational back-reaction effects, the evolution in quantum gravity might strongly differ from the semi-classical one.

In this article, we shall investigate the relationships between the choice of the dynamical framework and the description of processes. To this end, we shall compare the descriptions of thermal effects induced by uniform ac-

celeration, i.e. the Unruh[15] effect, in four dynamical frameworks. This comparative analysis clearly reveals the *generic* properties of the use of the BFA. (By generic we mean that these properties should also apply to quantum gravity.) In particular, it shows precisely when and why the description of processes obtained by using the BFA coincides to the description of the same processes in new frameworks in which the dynamics has been enlarged.

In order to perform this analysis, one should carefully choose the dynamical objects that shall be compared. We shall see that the comparison is the most instructive when focused on specific matrix elements controlling transition amplitudes[16]. Indeed these are the basic elements out of which all physical quantities can be computed. Furthermore they are very sensitive to the approximations used. Instead, integrated or averaged quantities, such as transition rates or the total energy emitted, are much less sensitive and therefore not appropriate to perform the comparison. Notice that the (relative) insensitivity of the transition rates is related to the thermodynamical nature of the Unruh effect.

The dynamical arena of the four frameworks we shall consider will be progressively enlarged and will incorporate the former one. Correspondingly, the description of the transitions will become more sophisticated. However, the enlargement of the dynamics shall provide new physical insights. In particular, the energy conservation law changes drastically when one abandons the BFA. In addition to this change which is not specific to the Unruh effect, there are also specific new informations obtained by the enlargement of the dynamics. The most interesting ones concern the strong relations between the Unruh effect and both pair creation in a constant electric field (in the third framework) and gravitational entropy (in the fourth one).

The first framework corresponds to the Unruh's original formulation of the problem[15]. In this case, one deals with a uniformly accelerated two-level atom coupled to a quantum massless field in Minkowski space time. The trajectory of the atom is treated at the BFA, i.e. it is once for all given and insensitive to the radiative events. Thus one can neither analyze the stability of the trajectory nor answer the question of the origin of the energy emitted by the accelerated system since momentum conservation is violated during the emission processes.

In the second framework, the position of the accelerated system is quantized. The particular model we shall consider consists of a "heavy" two-level ion immersed in a constant electric field and coupled, as before, to the

massless field[17][16]. In this second framework, the momentum and energy conservation laws are fully respected and one verifies that it is the electric field that provides the energy of the radiated quanta. We shall then establish under which conditions and for which reason the Unruh thermalization process is recovered in this new framework. The conditions are that both (i) the WKB approximation of the wave functions describing the system's position and (ii) a first order expansion in the energy changes induced by the transition should be valid. Then, Hamilton-Jacobi equations imply that the transition *rates* computed from both frameworks agree. (This agreement is absolutely generic in character: it also applies to quantum gravity when gravity plays the role of the heavy system[18].) This does not mean however that the transition *amplitudes* themselves coincide: there are indeed trans-Planckian phase shifts induced by exponentially large Doppler shifts. As a consequence, the mean flux radiated the accelerated system is smooth out by these recoil effects[16]. One thus realizes that the singular energy density obtained by treating the trajectory at the BFA[19][20] was an *artifact* directly induced by the use of this approximation which ignored recoils.

However this second framework is still an approximation since pair creation effects have not been taken into account. Indeed, since one is working with quantized relativistic fields, it is mandatory to work in second quantization. In this third framework, the new enlargement of the dynamics makes therefore contact with a an priori completely different phenomenon, namely pair creation in a constant electric field, i.e. the Schwinger effect[21]. It is remarkable that the amplitudes describing this latter effect are directly related to the radiative transitions by "crossing symmetry"[22], an intrinsic analytical property of amplitudes in QFT. Therefore, the Unruh effect and the Schwinger effect should be conceived as two aspects of a single dynamical theory, namely QFT in a constant electric field[23, 24]. In addition, this connection between pair creation amplitudes and radiative processes sheds light on the recently analyzed pair creation processes of charged black holes[25]-[29]. In particular, the role of the (gravitational) instanton which relates by tunneling process the Melvin geometry (in which the black holes are absent) to the Ernst geometry (present) is clarified[30].

This last remark naturally introduces the dynamical role of gravity that has been ignored so far. Indeed, upon turning on gravity, i.e. $G \neq 0$, the thermodynamical role of the acceleration horizon emerges. Then, the Unruh effect is included into the more general framework of thermodynamics of

horizons. In particular, the role of the horizon area acting as an entropy in delivering heat to accelerated systems (which are at rest with respect to the Killing horizon) is established. These relations between Euclidean gravity and thermal phenomena can also be conceived as particular examples of the thermodynamical approach to gravity[31] presented by Ted Jacobson.

In resume, by abandoning the BFA for describing the system's trajectory, we shall successively
1. restore energy-momentum conservation,
2. suppress singular behavior of the emitted fluxes,
3. relate radiative transitions to pair creation amplitudes and
4. connect thermal effects induced by constant acceleration to horizon entropy.

Throughout the text, we shall also mention what are the properties revealed by the analysis of Unruh's effect that are generic in character.

2 The transition amplitudes in the absence of recoils

In the original Unruh framework[15], the two level atom is maintained, for all times, on a single uniformly accelerated trajectory

$$t_a(\tau) = a^{-1}\mathrm{sinh}a\tau \ , \ z_a(\tau) = a^{-1}\mathrm{cosh}a\tau \tag{1}$$

a is the acceleration and τ is the proper time of the accelerated atom. We work for simplicity in Minkowski space time in $1+1$ dimensions. The two levels of the atom are designated by $|-\rangle$ and $|+\rangle$ for the ground state and the excited state respectively. The transitions from one state to another are induced by the operators $A, A^\dagger$

$$\begin{aligned} A|-\rangle = 0, \quad A|+\rangle = |-\rangle \\ A^\dagger|-\rangle = |+\rangle, \quad A^\dagger|+\rangle = 0 \end{aligned} \tag{2}$$

The atom is coupled to a massless scalar field $\phi(t,z)$. The Klein-Gordon equation is $(\partial_t^2 - \partial_z^2)\phi = 0$ and the general solution is thus

$$\phi(U,V) = \phi(U) + \phi(V) \tag{3}$$

where U, V are the light like coordinates given by $U = t - z, V = t + z$. The right moving part may be decomposed into plane waves:

$$\varphi_\omega(U) = \frac{e^{-i\omega U}}{\sqrt{4\pi\omega}} \tag{4}$$

where ω is the Minkowski energy. Using this basis, the Heisenberg field operator ϕ reads

$$\phi(U) = \int_0^\infty d\omega \left(a_\omega \varphi_\omega(U) + a_\omega^\dagger \varphi_\omega^*(U) \right) \tag{5}$$

where $a_\omega, a_\omega^\dagger$ are operators of destruction and creation of a Minkowski quantum of energy ω. The coupling between the atom and the field is taken to be, see [16],

$$\begin{aligned} H_{\rm int}(\tau) &= ga \left(A e^{-i\Delta m \tau} + A^\dagger e^{i\Delta m \tau} \right) \phi(U_a(\tau)) \\ &= ga \int_0^\infty d\omega \left[\left(A e^{-i\Delta m \tau} + A^\dagger e^{i\Delta m \tau} \right) \right. \\ &\qquad \left. \times \left(\frac{a_\omega e^{i\omega e^{-a\tau}/a}}{\sqrt{4\pi\omega}} + \frac{a_\omega^\dagger e^{-i\omega e^{-a\tau}/a}}{\sqrt{4\pi\omega}} \right) \right] \end{aligned} \tag{6}$$

where $U_a(\tau) = t_a(\tau) - z_a(\tau) = -e^{-a\tau}/a$. Because of the locality of the coupling, only $\phi(U_a(\tau))$ evaluated along the classical trajectory enters into the interaction.

The classically forbidden transition amplitude (spontaneous excitation) from the ground state $|-\rangle|0\rangle$, where $|0\rangle$ is Minkowski vacuum, to the excited state $|+\rangle|1_\omega\rangle$ containing one quantum of energy ω (where $|1_\omega\rangle = a_\omega^\dagger|0\rangle$) is

$$B(\omega, \Delta m, a) = \langle 1_\omega | \langle + | e^{-i\int d\tau H_{\rm int}} | - \rangle | 0 \rangle \tag{7}$$

To first order in g, one finds

$$\begin{aligned} B(\omega, \Delta m, a) &= -iga \int_{-\infty}^{+\infty} d\tau \; e^{i\Delta m \tau} \; \frac{e^{-i\omega e^{-a\tau}/a}}{\sqrt{4\pi\omega}} \\ &= ig \; \Gamma(-i\Delta m/a) \; \frac{(\omega/a)^{i\Delta m/a}}{\sqrt{4\pi\omega}} \; e^{-\pi\Delta m/2a} \end{aligned} \tag{8}$$

where $\Gamma(x)$ is the Euler function. This transition amplitude is closely related to the β coefficient of the Bogoliubov transformation[32] which relates

the Minkowski operators a_ω to the Rindler operators associated with the eigenmodes of $-i\partial_\tau$ (hence given by $\varphi_{Rindler}(\tau) = e^{-i\lambda\tau}$). This provides the dynamical justification of studying Bogoliubov coefficients which can be defined in the absence of any coupling to accelerated system. Their study is thus preparatory in character, exactly like the analysis of Green functions in free field theory. This point of view has been developed in [18] to incorporate gravitational "recoil" effects in quantum cosmology along the same lines that recoil effects of accelerated systems shall be treated in the next Section.

The transition amplitude for the inverse process, i.e. disintegration from $|+\rangle|0\rangle$ to $|-\rangle|1_\omega\rangle$, is given by $A(\omega, \Delta m, a) = B(\omega, -\Delta m, a)$. From eq. (8), one easily finds

$$A(\omega, \Delta m, a) = -B^*(\omega, \Delta m, a)\ e^{\pi \Delta m/a} \tag{9}$$

Thus for all ω one has

$$|\frac{B(\omega, \Delta m, a)}{A(\omega, \Delta m, a)}|^2 = e^{-2\pi\Delta m/a} \tag{10}$$

Since this ratio is independent of ω, the ratio of the probabilities of transitions (excitation and disintegration) is also given by eq. (10). Hence at equilibrium, the ratio of the probabilities P_-, P_+ to find the atom in the ground or excited state satisfy

$$\frac{P_+}{P_-} = |\frac{B(\omega, \Delta m, a)}{A(\omega, \Delta m, a)}|^2 = e^{-2\pi\Delta m/a} \tag{11}$$

This is the Unruh effect[15]: at equilibrium, the probabilities of occupation are thermally distributed with temperature $a/2\pi$.

Using the amplitudes $B(\omega, \Delta m, a)$ and $A(\omega, \Delta m, a)$, one can compute, to order g^2, the mean value of the flux emitted by the two level atom, see [33]-[36]. The point we wish to emphasize is that all the emitted Minkowski quanta, whatever is their energy ω, interfere so as to deliver a negative mean flux, for $U < 0$, whose interpretation is that some Rindler quantum has been absorbed[34]. However, the energy density is positive and singular on the horizon $U = 0$[19]. This singular and physically pathological behavior arises from the extremely well tuned phases of $B(\omega, \Delta m, a)$ and $A(\omega, \Delta m, a)$ for $\omega \to \infty$. These phases will be inevitably washed out after a finite proper time when recoils will be taken into account[16]. Then, the new value of the flux will be will be rapidly positive and perfectly regular around and on $U = 0$.

3 The amplitudes in first quantization

In this section, we first introduce the model of [17] which is similar to the one used by Bell and Leinaas[37]. This will allow us to take into account the momentum transfers to the accelerated system which are caused by the radiative emission processes.

We shall then establish when and why the thermalization of the inner degrees of freedom of the accelerated system is recovered. In particular, we shall see that the thermalization does not require a well defined classical trajectory; therefore it neither requires well defined "Rindler" energies nor a well defined location of the horizon. Indeed, even when one deals with delocalized waves, the thermal equilibrium ratio eq. (11) is obtained.

The model consist on two scalar charged fields (ψ_M and ψ_m) of slightly different masses (M and m) which will play the role of the former states of the atom: $|+\rangle$ and $|-\rangle$. The quanta of these fields are accelerated by an external classical constant electric field E with a given by

$$\frac{E}{M} = a \simeq \frac{E}{m} \tag{12}$$

because one imposes

$$\Delta m = M - m \ll M \tag{13}$$

to have the "light" mass gap Δm well separated from the "heavy" rest mass of the ion.

We work in the homogeneous gauge ($A_t = 0$, $A_z = -Et$). In that gauge, the momentum k is a conserved quantity and the energy p of a relativistic particle of mass M is given by the mass shell constraint $(p_\mu - A_\mu)^2 = M^2$:

$$p^2(M,k,t) = M^2 + (k + Et)^2 \tag{14}$$

The classical equations of motion are easily obtained from this equation and are given in terms of the proper time τ by

$$\begin{aligned} p(M,k,t) &= M \text{cosh} a\tau \\ t + k/E &= (1/a)\text{sinh} a\tau \\ z - z_0 &= (1/a)\text{cosh} a\tau \end{aligned} \tag{15}$$

Thus at fixed k, the time of the turning point (i.e. $dt/d\tau = 1$) is fixed whereas its position is arbitrary and given by $z_0 + 1/a$.

From eq. (14), the Klein Gordon equation for a mode $\psi_{k,M}(t,z) = e^{ikz}\chi_{k,M}(t)$ is

$$\left[\partial_t^2 + M^2 + (k+Et)^2\right]\chi_{k,M}(t) = 0 \tag{16}$$

When $\Delta m \simeq a$ and when eq. (13) is satisfied, one has $M^2/E \gg 1$. Then pair production amplitudes[21] may be safely ignored since the mean density of produced pairs scales like $e^{-\pi M^2/E}$, see next Section. Furthermore, the WKB approximation for the modes $\chi_{k,M}(t)$ is valid for all t. Indeed, the corrections to this approximation are smaller than $(M^2/E)^{-1}$. The modes $\psi_{k,M}(t,z)$ can be thus correctly approximated by

$$\psi_{k,M}^{WKB}(t,z) = \frac{e^{ikz}}{\sqrt{2\pi}}\frac{e^{-i\int^t p(M,k,t')dt'}}{\sqrt{p(M,k,t)}} \tag{17}$$

where $p(M,k,t')$ is the classical energy at fixed k given in eq. (14).

As emphasized in refs. [17][38], the gaussian wave packets in k do not spread if their width is of the order of $E^{1/2}$. In that case, the spread in z at fixed t is of the order of $E^{-1/2} = (Ma)^{-1/2}$ and thus much smaller than the acceleration length $1/a$. Since the stationary phase condition of the $\psi_{k,M}^{WKB}(t,z)$ modes gives back the accelerated trajectory and since the wave packets do not spread, one has constructed a quantized version of the accelerated system which tends uniformly to the BFA model when $M \to \infty, E \to \infty$ with $E/M = a$ fixed.

The interacting Hamiltonian which induces transitions between the quanta of mass M and m by the emission or absorption of a massless neutral quantum of the ϕ field is simply

$$H_{\psi\phi} = \tilde{g}M^2 \int dz \left[\psi_M^\dagger(t,z)\psi_m(t,z) + \psi_M(t,z)\psi_m^\dagger(t,z)\right]\phi(t,z) \tag{18}$$

where $\tilde{g}$ is dimensionless. In momentum representation, by limiting ourselves to the right moving modes of the ϕ field, one obtains

$$\begin{aligned} H_{\psi\phi} \;=\; & \tilde{g}M^2 \int_{-\infty}^{+\infty} dk \int_0^\infty \frac{d\omega}{\sqrt{4\pi\omega}} \\ & \times \Big\{ \left[b_{M,k-\omega}\chi_{M,k-\omega}(t)\; b_{m,k}^\dagger \chi_{m,k}^*(t) + \text{h.c.}\right] a_\omega e^{-i\omega t} \\ & + \left[b_{M,k+\omega}\chi_{M,k+\omega}(t)\; b_{m,k}^\dagger \chi_{m,k}^*(t) + \text{h.c.}\right] a_\omega^\dagger e^{+i\omega t} \Big\} \end{aligned} \tag{19}$$

where the operator $b_{M,k-\omega}$ detroys a quantum of mass M and momentum $k-\omega$ and the operator $b^\dagger_{m,k}$ creates a quantum of mass m and momentum k. Therefore the product $b_{M,k-\omega}b^\dagger_{m,k}$ plays the role of the operator A in eq. (6). However, the interaction between the radiation field ϕ and the two levels of the accelerated system is no longer a priori restricted to a classical trajectory. In the present context of homogeneous electric field, this leads to an exact conservation of momentum. Notice that we have not introduced anti-ion creation operators in $H_{\psi\phi}$. This is a legitimate truncation when working with WKB waves. This condition shall be relaxed in the next Section.

We first establish when and why the behavior of the $\chi_M(t)$ modes re-delivers the Unruh effect[16]. To this end, we shall not consider well localized wave packets. We emphasize this point. An alternative route, a priori equally valid, would consist in *first* making well localized wave packets and only *then* computing transition amplitudes, see [17]. It turns out that the averaging over k in constructing wave packets is both unnecessary and a nuisance in that it erases the detailed mechanisms that ensure thermalization in this new framework.

Thus, we focus on the transition amplitude at fixed momentum k that replaces $B(\Delta m, \omega, a)$, eq. (8). In the present context, it corresponds to the amplitude to jump from the state $|1_k, m\rangle|0, M\rangle|0\rangle$ to the state $|0, m\rangle|1_{k'}, M\rangle|1_\omega\rangle$. $|0, M\rangle$ designates the vacuum state for the ψ_M field and $|1_k, m\rangle = b^\dagger_{m,k}|0, m\rangle$ is the one particle state of the ψ_m field of momentum k. Due to momentum conservation, this amplitude can be expressed as

$$\begin{aligned}\delta(k \quad - \quad & k' - \omega)\ \tilde{B}(M, m, k, \omega) \\ = \quad & \langle 1_\omega|\langle 1_{k'}, M|\langle 0, m|e^{-i\int dt H_{\psi\phi}}|1_k, m\rangle|0, M\rangle|0\rangle \end{aligned} \tag{20}$$

compare this expression with eq. (7).
To first order in $\tilde{g}$, one finds

$$\begin{aligned}\tilde{B}(M, m, k, \omega) &= -i\tilde{g}M^2 \int_{-\infty}^{+\infty} dt\ \chi^*_{M,k-\omega}(t)\ \chi_{m,k}(t) \frac{e^{i\omega t}}{\sqrt{4\pi\omega}} \\ &= -i\tilde{g}M^2 \int_{-\infty}^{+\infty} dt \frac{e^{i\int^t dt' [p(M,k-\omega,t') - p(m,k,t')]}}{\sqrt{p(M, k-\omega, t)p(m, k, t)}} \frac{e^{i\omega t}}{\sqrt{4\pi\omega}} \end{aligned} \tag{21}$$

In the second line, we have used the WKB approximation eq. (17) for the χ modes.

As such, $\tilde{B}(M, m, k, \omega)$ seems very different from the BFA amplitude $B(\Delta m, \omega, a)$ given in eq. (8). Indeed, this latter expression was based on a well defined classical trajectory parametrized by τ whereas in the present case, it is momentum conservation that has been exactly taken into account.

However, it is precisely this conservation law that re-introduces the notion of classical trajectory (exactly like in quantum gravity the Wheeler-DeWitt constraint re-introduces the notion of time[18]). To explicitize this, we develop the phase and the norm of the *integrand* of eq. (21) in powers of ω and Δm. To first order in Δm and ω, the phase $\varphi(t)$ is *equal*, up to a constant, to the phase of the integrand of eq. (8). Indeed, one has

$$\begin{aligned} \varphi(t) &= \int^t dt' \left[p(M, k-\omega, t') - p(m, k, t') \right] + \omega t \\ &\simeq \Delta m \, \partial_M \int^t dt' p(M, k, t') - \omega \, \partial_k \int^t dt' p(M, k, t') + \omega t \\ &\simeq \Delta m \Delta \tau(t) - \omega (\Delta z_k(t) - t) \end{aligned} \tag{22}$$

$\Delta\tau(t)$ and $\Delta z_k(t)$ are the lapses of proper time and of space evaluated along a uniformly accelerated trajectory characterized by k. These relations may be checked explicitly by computing the integrals using eq. (14).

However, for establishing their universal validity, it is appropriate to realize that those relations are nothing but Hamilton-Jacobi relations. Indeed, these are

$$\partial_M S_{cl.}(M, k, t) = \partial_M \int^t dt' p(M, k, t') = \Delta\tau(t) \tag{23}$$

$$\partial_k S_{cl.}(M, k, t) = \partial_k \int^t dt' p(M, k, t') = \Delta z_k(t) \tag{24}$$

Thus, whatever is the nature of the external field which brings the system into constant acceleration, the first two terms of eq. (22) will always be found. More importantly and more generally, whatever is the problem one is considering, upon using WKB wave functions for describing “heavy” degrees of freedom, the transition amplitudes among “light” degrees will agree with the BFA expressions upon developing the amplitudes to first order in the light changes. Indeed, in that approximation, one must also neglect the dependence in ω and Δm in the denominator of eq. (21). Then the measure is $dt/p(M, k, t) = d\tau/M$. Thus, one has, as announced,

$$\tilde{B}(M, m, k, \omega) = -i\tilde{g}M \int_{-\infty}^{+\infty} d\tau e^{i\Delta m\tau} \frac{e^{-i\omega e^{-a\tau}/a}}{\sqrt{4\pi\omega}} \times \left(e^{-i\omega k/E} \right)$$

$$= \left[\frac{\tilde{g}M}{ga}\right] B(\Delta m, \omega, a) \ \times \left(e^{-i\omega k/E}\right) \tag{25}$$

Very important is the fact that the momentum k introduces only a phase shift with respect to the BFA amplitude $B(\Delta m, \omega, a)$. Thus any normalized superposition of modes $\psi_{m,k}$ specifying the initial wave function of the "heavy" system will give rise to the same probability to emit of photon of energy ω. Moreover the ratio of the square of $\tilde{B}(M, m, k, \omega)$ and $\tilde{A}(M, m, k, \omega)$ will necessarily satisfy eq. (10). Therefore, under these approximations, the two level ion thermalizes exactly as in the no-recoil case. Moreover, the total energy emitted by the "recoiling" ion also equals the corresponding energy evaluated at the BFA. The reason is simply that the total energy is a function of the norm of $B(M, m, \omega, k)$ only[16].

The lesson of this comparison of amplitudes is the following. When both the WKB approximation for the "heavy" system and a first order expansion in the light changes are valid, Hamilton-Jacobi equations guarantee that the norm of the transition amplitudes agree. Therefore all physical quantities that are functions of these norms only will automatically agree as well.

However, there are both conceptual and numerical differences between the amplitudes computed in the two frameworks. For instance the phase of the amplitudes do not agree even to first order in ω and Δm. We shall illustrate these differences by first considering the stationary phase condition in both cases and then by analyzing the consequences of the phase shifts in the determination of the flux emitted by the atom.

In this present framework, the stationary point t^* of the integrand of eq. (21) is at

$$p(M, k - \omega, t^*) - p(m, k, t^*) + \omega = 0 \tag{26}$$

This is the conservation law of the Minkowski energy. Instead, in the Unruh framework, from the first line of eq. (8), one finds

$$\Delta m + \omega e^{-a\tau} = 0 \tag{27}$$

which is the resonance condition in the accelerated frame, i.e. conservation of Rindler energy. This difference also arise in gravitational situations, see [18]: Upon dealing with gravity described at the BFA, one finds that matter processes satisfy energy conservation in the given background; this is the equivalent of eq. (27). And upon abandoning the BFA and solving twice

Einstein's equations, one verifies that the resonance condition involves the energy of gravity, exactly like eq. (26) contains the difference of two heavy energy $p(M, k, t)$.

Of course, the two versions of energy conservation must coincide in the limit $\Delta m/M \to 0$. Indeed, by taking the square of eq. (26) and using eq. (14) one gets

$$\frac{M^2 - m^2}{2} = \omega \left[(k - \omega + Et^*) - p(M, k - \omega, t^*) \right] \tag{28}$$

Introducing once more the proper time of the heavy ion M, eq. (15), one finds

$$\Delta m (1 - \Delta m/2M) = -\omega e^{-a\tau^*} \qquad \text{QED} \tag{29}$$

The second point we wish to make is the following. There is a strict relation between conservation of momentum, eq. (20), and energy, eq. (26), and the modifications of the properties of the emitted fluxes, see [16] for more details. Indeed, the two level ion constantly loses energy and momentum in accordance with these conservation laws. Then the trajectory of the ion drifts from orbits to neighboring ones in t and z corresponding to laters times and greater z. The total change in the time of the turning point is $E\Delta t_{t.p.} = \sum_i \omega_i$, i.e. it is proportional to the total momentum lost. One also verifies that the total change in position is $\sum_i \omega_i/E$. These successive changes of hyperbolae lead to the decoherence of the emissions causing these changes since the very peculiar phases obtained in the BFA framework are washed out by the recoils. Then, both "unphysical" properties obtained in that framework, namely negativity of the energy density during arbitrary large proper times[34]-[36] and singular behavior on the horizon[19][20], are eliminated.

There is a third point that can be addressed before considering corrections to the WKB approximation. It concerns the possibility in computing corrections of the equilibrium ratio, eq. (11), induced by higher order terms in $\Delta m/M$. This problem shall be discussed elsewhere.

4 The amplitudes in second quantization

In the former Section, we used WKB wave functions which are valid approximate solutions of eq. (16) when $M^2/E \gg 1$. Then, positive and negative

energy solutions completely decouple. This is no longer true when one deals with the exact solutions of eq. (16). In this new case indeed, one must introduce two sets of modes $(\chi_M^{in}, \chi_M^{out})$ which only asymptotically, i.e. for $t \to \pm\infty$, define particle states. That the two states do not coincide (they are related by a linear (Bogoliubov) transformation) indicates that the initial vacuum $|0, M, in\rangle$ associated with the initial set χ_M^{in} spontaneously decays and that particles will be find at $t \to \infty$. In the present case of a constant electric field, one finds that the mean number of quanta of momentum k is given by

$$N_M = |\beta_M|^2 = e^{-\pi M^2/E} \tag{30}$$

where β_M is the Bogoliubov coefficient, i.e. the amount of negative energy *out* solution in a purely positive *in* solution, see *e.g.* [39].

Using Feynman rules, one can re-calculate the ratio of the transition rates to emit a photon starting from the ground state (m) and from the excited state (M) by taken into account vacuum instability with respect to pair creation of both ion fields, see Nikishov[23]. This ratio turns out to be intimately related[17][22] to eq. (30) in that it is equal to

$$|\frac{\mathcal{B}(M,m,p,\omega)}{\mathcal{A}(M,m,p,\omega)}|^2 = e^{-\pi(M^2-m^2)/E} = \frac{N_M}{N_m} \tag{31}$$

To re-calculate this ratio, we analyze the amplitude $\mathcal{B}(M,m,k,\omega)$ to emit a massless quantum of energy ω starting from the state m. This amplitude strictly corresponds to the amplitude $\tilde{B}(M,m,k,\omega)$ of eq. (20). To first order in $\tilde{g}$, it is given by

$$\begin{aligned}
\delta(k-k'-\omega)&\mathcal{B}(M,m,k,\omega) \\
&= \langle 1_\omega|\langle 1_{k'}, M, out|\langle 0, m, out|e^{-i\int dt H_{\psi\phi}}|1_k, m, in\rangle|0, M, in\rangle|0\rangle \\
&= -\delta(k-k'-\omega)\ i\tilde{g}M^2\left(Z_M Z_m \alpha_M^{-1}\alpha_m^{-1}\right) \\
&\qquad \times \int_{-\infty}^{\infty} dt\ \chi_{M,k-\omega}^{in*}(t)\ \chi_{m,k}^{out}(t)\ \frac{e^{i\omega t}}{\sqrt{4\pi\omega}}
\end{aligned} \tag{32}$$

The factor $Z_M Z_m$ is the product of the overlaps of the *in* and *out* vacuum states of both charged fields. $\alpha_{M(m)}$ is a coefficient whose norm is given by $\sqrt{1 + |\beta_{M(m)}|^2}$. These factors take into account the fact that the scattering process happens in the presence of pair production of charged quanta. They all reduce to one in the WKB limit, $M^2/E \to \infty$.

This amplitude can be exactly evaluated in terms of the integral representations of the χ modes, see [22]. One obtains,

$$\begin{aligned}\mathcal{B}(M,m,k,\omega) &= -i\frac{\tilde{g}M^2}{2E}\,\Gamma(-i(M^2-m^2)/2E) \\ &\quad\times\frac{(\omega)^{i(M^2-m^2)/2E}}{\sqrt{4\pi\omega}}\,e^{-\pi(M^2-m^2)/2E} \\ &\quad\times e^{i(\omega p-\omega^2/2)/E}\,\left[\Gamma(i\frac{M^2}{2E}+\frac{1}{2})\frac{e^{\pi M^2/2E}(E/2)^{-iM^2/2E}}{\sqrt{2\pi}}\right]\end{aligned} \tag{33}$$

This expression should be compared with eq. (8). As in that case, the determination of the equilibrium population requires only to know the ratio of the transition rates. And as in that case, this ratio can be obtained effortless from the analytical behavior of the amplitude in ω, since $\mathcal{A}^*(M,m,k,\omega) = \mathcal{B}(M,m,k,e^{-i\pi}\omega)$. (Notice that this analytical behavior in ω encodes the stability condition of the Minkowski vacuum of the radiation field. Notice also that Hawking radiation can be derived in similar terms[15].) From this relation and eq. (33), one immediately deduces that the ratio of the transition rates satisfy eq. (31).

Therefore we have proven that the equilibrium probabilities defined by radiative processes are equal to those defined by the Schwinger process. However, the direct proof that both processes are intimately related follows from the fact that their respective amplitudes are interchanged by crossing symmetry, see [22]. Indeed what corresponds to a pair creation diagram in the direct channel, describes a radiative transition in the "crossed" channel.

To conclude this Section, we analyze the relationships between this formulation of radiative processes with the former ones studied in Section 2 and 3. First notice that in the limit $M^2/E \to \infty$ at fixed $M - m = \Delta m$ and $M/E = 1/a$, the *integrand* of eq. (32) tends uniformly to the *integrand* based on WKB expressions in eq. (21). This indicates once more that the rates agree because matrix elements defining transition amplitudes can be put in strict correspondence, at fixed quantum number.

Secondly, it should not have escaped the reader that eq. (31) differs from the Boltzmanian ratio found using the BFA, i.e. eq. (10). Indeed, in order to make contact with this thermal *canonical* distribution, one must consider the limit $\Delta m \ll M$. Then, in the present second quantized framework, the concepts of acceleration and temperature are brought to bear for the first

time. They both appear through a first order change in Δm. This emergence of classical concepts bears many similarities with statistical mechanics since it is also through a first order change in the energy that the concept of temperature arises from *microcanonical* ensembles.

To bring about this contact with statistical mechanics, it is most instructive to use again Hamilton-Jacobi equations but applied this time to Euclidean dynamics. Indeed relationships with both eq. (23) and black hole thermodynamics will become clear. We remind the reader that the Schwinger pair creation amplitudes may be understood from the Euclideanized version of the problem. In fact, it is $S_{euclid} = \pi M^2/E$, the classical action to complete a Euclidean (closed) orbit, which determines their rate, see eq. (30). From this action, one computes the Euclidean proper time necessary to complete this orbit. It is given by

$$\tau_{euclid} = \partial_M S_{euclid} = \partial_M \left(\frac{\pi M^2}{E} \right) = \left(\frac{a}{2\pi} \right)^{-1} \tag{34}$$

It equals the inverse Unruh temperature. Thus, to first order in Δm, it is meaningful to write eq. (31) as

$$|\frac{\mathcal{B}(M,m,k,\omega)}{\mathcal{A}(M,m,k,\omega)}|^2 \simeq e^{-\Delta m \partial_M S_{euclid}} = e^{-\Delta m 2\pi/a} = |\frac{B(\Delta m,a,\omega)}{A(\Delta m,a,\omega)}|^2 \tag{35}$$

This shows that the Unruh process can be viewed as an infinitesimal ratio of two Schwinger processes. Moreover the instanton action S_{euclid} acts as an *entropy* in delivering the Unruh temperature. Indeed,

$$\tau_{euclid} \Delta m = \Delta S_{euclid} \tag{36}$$

is the first law of thermodynamics[1].

In view of the similarities between Unruh effect and black hole radiation, it is inviting to inquire about the relationship between this "instantonic" entropy and the gravitational entropy of black holes whose variation determines Hawking temperature. This is the subject of next Section.

[1] For the skeptical reader, we add that this analysis has been enlarged[24] by replacing the massless neutral field by a charged field. Then the new equilibrium ratio is determined by an extended thermodynamical relation in which the work done by the electric field contributes as well. The analogy with the thermodynamics of charged black holes is manifest.

5 The amplitudes in the presence of gravity

This short Section is of a heuristic character and the level of mathematical rigor will be lowered simply because Quantum Gravity does not exist. Indeed, to describe quantum transitions in which gravity does play an active role, i.e. in which the BFA for gravity has been abandoned, new approximations should be adopted. The type of approximations we shall need and use are quite similar to those we used in Section 3 for describing the system's trajectory: Gravitational quantum effects will approximatively be taken into account like momentum recoils were taken into account in eq. (21), i.e. in matrix elements describing transition amplitudes of light (matter) degrees of freedom, the initial and final wave functions of the heavy system (gravity) bring their own classical action, see [18].

To illustrate these aspects, we start the discussion by recalling the results of [25]-[29]. In these references, the probability of creation charged black holes in a constant electric field was estimated by making use of two hypothesis. First the authors required that the Euclidean manifold describing the instanton responsible for the creation be regular. Secondly they assumed that the probability of creation depends on the instanton action computed from the Hilbert-Einstein like S_{euclid}, the classical action to complete a Euclidean orbit, determined the production rate in eq. (30). Using these hypothesis, they found that the probability to produce a pair of black holes characterized by the mass M and charge Q is

$$P_{M,Q} = e^{\delta \mathcal{A}_{acc}/4G} \times e^{\mathcal{A}_{BH}/4G} \tag{37}$$

$\mathcal{A}_{BH}(M,Q)$ is the area of the black hole horizon. In this expression, $e^{\mathcal{A}_{BH}/4G}$ furnishes the density of black hole states with mass M and charge Q thereby confirming the Bekenstein interpretation of $\mathcal{A}_{BH}/4G = S_{BH}$ as being the black hole entropy.

$\delta\mathcal{A}_{acc}(M,Q,E)$ is the *change* of the area of the acceleration horizon induced by the creation of the black hole pair[40].

The domain of the one parameter family (i.e. the values of M and Q which satisfy the regularity condition) which can be compared with the Schwinger mechanism is the one in which the black holes radius is much smaller than the inverse acceleration, i.e. in the point particle limit. This limit corresponds

to the limit $G \to 0$. In that case, one finds

$$\frac{\delta \mathcal{A}_{acc}}{4G} = -\pi M^2/QE = -S_{euclid} \tag{38}$$

This is the usual instanton action to create charged particles.

First notice that it is independent of G, as it should be. Indeed, the limit $G \to 0$ in the present gravitational case strictly corresponds to the limit $\Delta m/M \to 0$ in Sections 3 and 4. In that case, we saw that the transition probabilities coincide with the BFA probabilities since the integrands of the amplitudes leading to these probabilities agree up to a phase. Similarly, by turning on gravity and then taking the limit $G \to 0$, one must recover the BFA probabilities. That this is the case, justify a posteriori the assumptions used in [29].

Secondly, eq. (38) and eq. (36) show that S_{euclid} acts as a gravitational entropy. Indeed, eq. (36) can be rewritten as

$$\frac{2\pi}{\kappa}\Delta m = \frac{\Delta \mathcal{A}_{acc}}{4G} \tag{39}$$

where $\kappa = a$ is the surface gravity of the accelerating horizon measured along the trajectory $z^2 - t^2 = 1/a^2$. $\Delta \mathcal{A}_{acc}$ is the change of the accelerating horizon area associated with the replacement of the heavy mass M by the lighter one m[30]. This rewriting strongly suggests that changes in area divided by $4G$ determine equilibrium distributions of accelerated systems according to the first law of horizon thermodynamics.

More details on these aspects can be found in [30][24].

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References

[1] W. G. Anderson, P. R. Brady, Werner Israel, and S. M. Morsink, Phys. Rev. Lett. 70 (1993) 1041

[2] L. M. Burko and A. Ori, Phys. Rev. Lett. 74 (1995) 1064

[3] S. P. Trivedi, "Semiclassical extremal black holes", hep-th/9211011

[4] T. Jacobson, "Semiclassical decay of near extremal black holes", hep-th/9705017

[5] S. W. Hawking, Commun. Math. Phys. **43** (1975) 199

[6] J. M. Bardeen, Phys. Rev. Lett. **46** (1981) 382

[7] R. Parentani and T. Piran, Phys. Rev. Lett. **73** (1994) 2805

[8] S. Massar, Phys. Rev. D **52** (1995) 5861

[9] G. 't Hooft, Nucl. Phys. B **256** (1985) 727

[10] T. Jacobson, Phys. Rev. D **44** (1991) 1731 and **48** (1993) 728

[11] W. G. Unruh, Phys. Rev. D **51** (1995) 2827

[12] R. Brout, S. Massar, R. Parentani and Ph. Spindel, Phys. Rev. D **52** (1995) 4559

[13] S. Massar and R. Parentani, Phys. Rev. D **54** (1996) 7431

[14] Y. Kiem, H. Verlinde, E. Verlinde *Quantum Horizons and Complementarity.* CERN-TH-7469-94, hep-th/9502074

[15] W. G. Unruh, Phys. Rev. D **14** (1976) 870

[16] R. Parentani, Nucl. Phys. B **454** (1995) 227

[17] R. Brout, R. Parentani and Ph. Spindel, Nucl. Phys. B **353** (1991) 209

[18] R. Parentani, Nucl. Phys. B. **492** (1997) 475, 501, gr-qc/9610044 and 45.

[19] W. G. Unruh, Phys. Rev. D **46** (1992) 3271

[20] S. Massar and R. Parentani, Phys. Rev. D **54** (1996) 7426

[21] J. Schwinger, Phys. Rev. **82** (1951) 664

[22] S. Massar and R. Parentani, Phys. Rev. D **55** (1997) 3603

[23] A. I. Nikishov, Sov. Phys. JETP **30** (1970) 660

[24] C. Gabriel, Ph. Spindel, S. Massar, R. Parentani, "Interacting Charged Particles in an Electric Field and the Unruh Effect", hep-th/9706030

[25] G. W. Gibbons, in "Fields and Geometry", Proceedings of the 22nd Karpacz Winter School of Theoretical Physics, Karpacz, Poland, 1986, edited by A. Jadczyk (World Scientific, Singapore, 1986)

[26] D. Garfinkle and A. Strominger, Phys. Lett. B **256** (1991) 146

[27] S. W. Hawking and S. Ross, Phys. Rev. D **52** (1995) 5865

[28] F. Dowker, J. P. Gauntlett, S. B. Giddings and G. T. Horowitz, Phys. Rev. D **50** (1994) 2662

[29] S. W. Hawking, G.T. Horowitz and S. Ross, Phys. Rev. D **51** (1995) 4302

[30] S. Massar and R. Parentani, Phys. Rev. Lett. **78** (1997) 3810

[31] T. Jacobson, Phys. Rev. Lett. **75** (1995) 1260

[32] S. A. Fulling, Phys. Rev. D **7** (1973) 2850

[33] W. G. Unruh and R. M. Wald, Phys. Rev. D **29** (1984) 1047

[34] P. Grove, Class. Quant. Grav. **3** (1986) 801

[35] D. Raine, D. Sciama and P. Grove, Proc. R. Soc. A **435** (1991) 205

[36] S. Massar, R. Parentani and R. Brout, Class. Quant. Grav. **10** (1993) 385

[37] J. S. Bell and J. M. Leinaas, Nucl. Phys. B **212** (1983) 131

[38] R. Brout, S. Massar, R. Parentani, S. Popescu and Ph. Spindel, Phys. Rev. D **52** (1995) 1119

[39] R. Brout, S. Massar, R. Parentani and Ph. Spindel, Phys. Rep. **260** (1995) 329

[40] S. W. Hawking and G.T. Horowitz, Class. Q. Grav. **13** (1996) 1487